Outline of solutions to Exam January 2012 in Financial Econometrics A. **Anders Rahbek, Copenhagen.**

Question A:

Question A.1: A GARCH(1,1) to fit the data? No - non-constant parameters, and misspecified - would expect $\alpha + \beta = 1$ and the LM misspecification tests confirm misspecified model. That the ARCH effects are removed by the GARCH(1,1) is to be expected similar to the discussions in lectures about the GARCH(1,1) as a filter.

Question A.2: With, $L_T(\theta) = \sum_{t=1}^{T} \left(-\frac{1}{2} \log \sigma_t^2 - \frac{y_t^2}{2\sigma_t^2} \right)$, we find immediately, with $z_t^2 = \frac{y_t^2}{\sigma_t^2}$, $S_T(\theta_0) = \sum_{t=1}^{T} \frac{1}{2} \left(z_t^2 - 1 \right) w_t, \quad w_t = \frac{x_{t-1}^2}{\omega_0 + \alpha_{x,0} x_{t-1}^2 + \alpha_{y,0} y_{t-1}^2}$.

Question A.3: Using the CLT for MGD sequences (Lecture notes Part II, Theorem II.1) set:

$$X_t = (y_t, x_t)'$$

then the CLT gives the result provided X_t is stationary and weakly mixing. Moreover, we need

$$Ew_t^2 = E \left(\frac{x_{t-1}^2}{\omega_0 + \alpha_{x,0} x_{t-1}^2 + \alpha_{y,0} y_{t-1}^2} \right)^2 < \infty,$$

but this holds if just $\alpha_{x,0} > 0$ as $w_t^2 \leq 1/\alpha_{x,0}$.

We conclude that if $\alpha_{x,0} > 0$ and $X_t = (y_t, x_t)'$ stationary and weakly mixing it holds.

Standard application of Theorem III.2 in Part 3 of lecture notes give $\sqrt{T} \left(\hat{\alpha}_x - \alpha_{x,0} \right)$ asymptotically Gaussian provided also information and third derivative converges in addition to the score.

Question A.4:

Model is clearly well-specified - $\hat{\alpha}_x$ appears to be significantly different from zero. and normality and no-arch in residuals not rejected. However, the x_t factor looks not to be stationary – $\rho = 1$ in the AR(1) specification, or I(1) – in which case we cannot conclude anything based on the "std. errors" as we would not be able to use that $\hat{\theta}$ is asymp Gaussian. But we observe even if x_t is I(1) (or even explosive), Ew_t^2 is bounded, and we would conjecture (based upon results for ARCH(1)) that we may because of the shape of w_t, or rather, because of the model, actually apply the std. errors in the usual way despite the I(1)ness of x_t.

Question B:

Question B.1: Clearly a misspecified model - the smoothed residuals not normal, and "ARCH" effect indicate un-modelled correlation structures in σ_t^2. Note that 2-state chain is nearly absorbing. Also clear from Figure 2.1 that we need a model with time-varying parameters, so a MS model may be a good starting point.

Question B.2: As long as $p_1, p_2, p_3 > 0$ all states can be reached with positive probabilty - as s_t is iid, and is a "non-irreducible and non-periodic" Markov chain, s_t is stationary - and hence σ_{s_t} is, and also y_t : y_t inherits the properties of s_t (or rather σ_{s_t}) as z_t are iidN$(0,1)$. Draw u_t iid U[0,1], and set $s_t = 1$ if $u_t < p_1$, $s_t = 2$ if $p_1 < u_t \leq p_1 + p_2$ and $s_t = 3$ if $u_t > p_1 + p_2$. This way $P(s_t = 1) = p_1$, $P(s_t = 2) = p_2 + p_1 - p_1 = p_2$ and finally $P(s_t = 3) = 1 - (p_1 + p_2) = p_3$.

Question B.3:

$$L(Y, S; \theta) = \sum_{t=1}^{T} \log \left(\prod_{i=1}^{3} (p_i f_{\sigma_i}(y_t))^{1(s_t=i)} \right) = \sum_{t=1}^{T} \sum_{i=1}^{3} 1(s_t = i)(\log f_{\sigma_i}(y_t) + p_i) \tag{1}$$

$$L_{EM}(Y; \theta) = \sum_{t=1}^{T} \sum_{i=1}^{3} p_{i,t}^{*}\left(\tilde{\theta}\right)(\log f_{\sigma_i}(y_t) + p_i), \tag{2}$$

with

$$p_{i,t}^{*} = p_{i,t}^{*}\left(\tilde{\theta}\right) = P(s_t = i|Y) \tag{3}$$

In fact, due to the mixture properties of the model,

$$p_{i,t}^{*} = P(s_t = i|y_t) = \frac{P(s_t=i, y_t)}{P(y_t)} = \frac{p_i f_{\sigma_i}(y_t)}{\sum_{i=1}^{3} p_i f_{\sigma_i}(y_t)}. \tag{4}$$

Question B.4: Differentiating $L_{EM}(Y_T; \theta)$ wrt σ_1^2 one gets,

$$\partial L_{EM}/\partial\sigma_1^2 = \frac{\partial}{\partial\sigma_1^2} \sum_{t=1}^{T} \sum_{i=1}^{3} p_{i,t}^{*}\left(\tilde{\theta}\right)\left(-\tfrac{1}{2}\log\sigma_i^2 - \tfrac{y_t^2}{2\sigma_i^2} + p_i\right) \tag{5}$$

$$= \sum_{t=1}^{T} p_{1,t}^{*}\left(\tilde{\theta}\right)\left(-\tfrac{1}{2\sigma_1^2} + \tfrac{y_t^2}{2\sigma_1^4}\right) = 0 \tag{6}$$

which is solved by,

$$\hat{\sigma}_1^2 \sum_{t=1}^{T} p_{1,t}^{*}\left(\tilde{\theta}\right) y_t^2 = \sum_{t=1}^{T} p_{1,t}^{*}\left(\tilde{\theta}\right). \tag{7}$$

This is similar to WOLS, where the smooth probabilities play the role of weights.

Question B.5:

$$P\left(\sigma_t = \sigma^* \; | \sigma_{t-1} = \sigma_1\right) = P\left(s_t = 2, 3 | s_{t-1} = 1\right) = p_{12} + p_{13}$$

$$\begin{aligned}
&P\left(\sigma_t = \sigma^* | \sigma_{t-2} = \sigma_1\right) \\
&= P\left(s_t = 2, 3 | s_{t-2} = 1\right) \\
&= P\left(s_t = 2, 3, s_{t-1} = 1, 2, 3 | s_{t-2} = 1\right) \\
&= \sum_{i=1}^{3} P\left(s_t = 2, 3 | s_{t-1} = i\right) P\left(s_{t-1} = i | s_{t-2} = 1\right) \\
&= \sum_{i=1}^{3} \sum_{j=2}^{3} P\left(s_t = j | s_{t-1} = i\right) P\left(s_{t-1} = i | s_{t-2} = 1\right) = \sum_{i=1}^{3} \sum_{j=2}^{3} \left(p_{ij} p_{1i}\right) = \left(p_{12} p_{11} + p_{13} p_{11}\right) + ...
\end{aligned}$$

Would conclude that process y_t can spend "more time" (as when compared to the two-state MS) with same σ^2 despite changing states s. Also one could discuss if actually really a 2-state MS when $\sigma_2 = \sigma_3 = \sigma^*$ and the p_{ij} entries can be "reduced".